MATH 233 Recitation

All questions

Andy Badea and Connor Magoon

Problem 1. Let $\mathbf{v} = \langle 1, 0, 3 \rangle$ and $\mathbf{w} = \langle 0, -2, 4 \rangle$.

(a) Compute $\mathbf{v} \cdot \mathbf{w}$.

(b) What is the angle between $\mathbf{v}$ and $\mathbf{w}$?

(c) Let $\mathbf{z} = \mathbf{w} - \frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{v}|^2}\mathbf{v}$. Compute the components of $\mathbf{z}$.

(d) Compute $\mathbf{z} \cdot \mathbf{v}$ and $\mathbf{z} \cdot \mathbf{w}$.

(e) Compute $|\mathbf{z}|^2$.

(f) What is the angle between $\mathbf{z}$ and $\mathbf{v}$? What is the angle between $\mathbf{z}$ and $\mathbf{w}$?

(g) Use the answers above to compute $|\mathbf{z} + t\mathbf{w}|^2$ for any real number t.

Problem 2. For what value of a is the vector $\mathbf{v} = \langle 5, 6, -7\rangle$ perpendicular to $\mathbf{w} = \langle 4 - a, a, -1\rangle$.

Problem 3. Let the vectors $\mathbf{r} = \langle 1, 0, -2\rangle$ and $\mathbf{q} = \langle 0, 3, 4\rangle$, and let $\mathbf{p}(t) = \langle 1 + t, 4 - 3t, -1 + 2t\rangle$ be a vector depending on a real number t.

(a) Find the value of t for which the quantity $|\mathbf{p}(t) - \mathbf{r}|^2$ is minimized. Compute $\mathbf{p}(t)$ at this value.

(b) Find the value of t for which the quantity $|\mathbf{p}(t) - \mathbf{q}|^2$ is minimized. Compute $\mathbf{p}(t)$ at this value.

(c) Find the value of t for which the quantity $|\mathbf{p}(t) - \mathbf{r}|^2 + |\mathbf{p}(t) - \mathbf{q}|^2$ is minimized. Compute $\mathbf{p}(t)$ at this value.

Problem 4. For what value of t is the vector $\mathbf{v} = \langle 5, 6, -7 \rangle$ perpendicular to $\mathbf{w} = \langle 4 - t, t, -1 \rangle$? Check your result by verifying $\mathbf{v} \cdot \mathbf{w} = 0$.

Problem 5. Let $\mathbf{r} = \langle 1, 1 \rangle$ and let $\mathbf{p}(t) = \langle t, 2 + 2t \rangle$ be a vector depending on a real number t. As t changes, what type of curve does $\mathbf{p}(t)$ trace (draw)? Find the value of t for which the quantity $|\mathbf{p}(t) - \mathbf{r}|^2$ is minimized. Evaluate $\mathbf{p}(t)$ at this t. What is the interpretation of this point? Find this point again, by using projections.

Definition 1. Definition 2.8 in the book. When a constant force is applied to an object so the object moves in a straight line from point P to point Q, the work W done by the force $\mathbf{F}$, acting at an angle θ from the line of motion, is given by $W = \mathbf{F} \cdot \mathbf{PQ} = \|\mathbf{F}\| \|\mathbf{PQ}\| \cos\theta$.

Problem 6. First: what are the units or dimensions of W? Then, problem 176 in the book. A sled is pulled by exerting a force of 100N on a rope that makes an angle of 25° with the horizontal. Find the work done in pulling the sled 40m.

Problem 7. Determine if the following claims are true or false. Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be vectors in $\mathbb{R}^3$ (three dimensions). If false, give a simple example that fails.

(a) $\mathbf{u} \times \mathbf{u} = \mathbf{0}$.

(b) $\mathbf{u} \times \mathbf{v} = \mathbf{v} \times \mathbf{u}$.

(c) If $\mathbf{u}, \mathbf{v} \neq \mathbf{0}$ and $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ then $\mathbf{u} = \mathbf{v}$.

(d) $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$.

(e) $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \mathbf{u} \cdot (\mathbf{w} \times \mathbf{v})$

(f) $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \mathbf{v} \cdot (\mathbf{w} \times \mathbf{u}) = \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})$

(g) $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$

Problem 8. Compute $\mathbf{u} \times \mathbf{v}$ where $\mathbf{u} = \langle -1, 0, e^t \rangle$, $\mathbf{v} = \langle 1, e^{-t}, 0 \rangle$.

Problem 9. Compute $\mathbf{w} \cdot (\mathbf{v} \times \mathbf{u})$ and $\mathbf{u} \cdot (\mathbf{w} \times \mathbf{v})$ where $\mathbf{u} = \langle 4, 2, -1 \rangle$, $\mathbf{v} = \langle 2, 5, -3 \rangle$, and $\mathbf{w} = \langle 9, 5, -10 \rangle$.

Problem 10. What is the geometric meaning of a triple scalar product evaluating to zero? What does the triple scalar product measure? Let $\mathbf{u} = \langle 1, 4, -7 \rangle$, $\mathbf{v} = \langle 2, -1, 4 \rangle$, and $\mathbf{w} = \langle 0, -9, 18 \rangle$. Compute $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$.

Problem 11. The following question sets up a line, and computes the distance between a point and the line using the cross-product.

(a) Describe the line L going through the points $P(3, -1, 3)$ and $Q(7, 1, 4)$ parametrically with vectors.

(b) Theorem: Let L be a line in space passing through point P with direction vector $\mathbf{v}$. If M is any point not on L, then the distance from M to L is $d = \frac{\|\mathbf{PM} \times \mathbf{v}\|}{\|\mathbf{v}\|}$. Compute the distance of the point $M(1, 1, 3)$ to the line in (a).

Problem 12. Compute the equation of the plane normal to the vector $\langle 4, -3, 1\rangle$ which passes through the point $P(1, 0, 5)$.

Problem 13. Let $\mathbf{r}(t) = \langle \cos t \sin \frac{t}{2}, \sin t \sin \frac{t}{2}, \sqrt{3} \sin \frac{t}{2}\rangle$.

(a) Compute the derivative $\mathbf{r}'(t)$.

(b) Compute $|\mathbf{r}'(t)|$ and use this to compute the arc-length of the curve $\mathbf{r}(t)$ for $0 \leq t \leq 4\pi$.

(c) Compute the dot product $\mathbf{r}'(t) \cdot \mathbf{r}''(t)$.

Problem 14. Let $\mathbf{u}(t)$ be a vector valued function with the property that $\mathbf{u}''(t) = \mathbf{u}(t) \times \mathbf{u}'(t)$. By differentiating the quantity $|\mathbf{u}'(t)|^2$, show that the curve $\mathbf{u}(t)$ is parameterized at a constant speed. By differentiating the quantity $|\mathbf{u}(t)|^2$ twice, show that $|\mathbf{u}(t)| = \sqrt{at^2 + bt + c}$ for some constants a, b and c.

Problem 15. Sketch the level curves of the functions $f(x, y) = x^2 + y^2$, and $g(x, y) = xy - x$.

Problem 16. Compute the arc-length of the curve $\mathbf{r}(t) = \langle t, t^2, \frac{2}{3}t^3 \rangle$ from $0 \leq t \leq 3$.

Problem 17. Compute the equation of the plane normal to the vector $\langle 4, -3, 1 \rangle$ which passes through the point $P(1, 0, 5)$.

Problem 18. Compute the distance between the plane defined by the equation $x - 4y + 2z = 7$ and the point $Q(0, 3, 1)$.

Problem 19. Classify the quadric surface defined by $x^2 + z^2 + 4y = 0$ and draw the cross-sections through the surface in the xy, xz, yz coordinate planes.

Problem 20. Rewrite and then simplify the equation of the quadratic surface $x^2 + y^2 \pm 4z^2 = 16$ in cylindrical polar coordinates. Classify the quadric for each sign $\pm$ and draw the surface.

Problem 21. Let $x(t) = ae^{-t}\cos(t), y(t) = be^{-t}\sin(t), z(t) = t$ be a parametric curve. Draw the curve in 3D and its projection in 2D onto the xy plane. Compute the derivative of the vector-valued function $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ which points to the curve, $\frac{d}{dt}\mathbf{r}(t)$. What is the interpretation of this derivative? Evaluate the derivative at $t = 0$ and incorporate it into your drawings.

Problem 22. Compute the following limits or show that they do not exist. If they do not exist, check whether the limit along different curves exists and differs by the curve.

(a) $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$

(b) $\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2}$

(c) $\lim_{(x,y,z)\to(0,0,0)} \frac{x^2-y^2-z^2}{x^2+y^2-z^2}$

(d) $\lim_{(x,y)\to(0,0)} \ln(x+y)$

Problem 23. Let $z = f(x, y) = \sin(3x)\cos(3y)$. Compute $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

Problem 24. Let $z = f(x, y) = \exp^{xy} \sin(x)\cos(y)$. Compute $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

Problem 25. Show that $z = e^{-t}\cos\left(\frac{x}{c}\right)$ satisfies the heat equation $\frac{\partial z}{\partial t} = c^2 \frac{\partial^2 z}{\partial x^2}$ (compute the derivatives and check if the equality is true)

Problem 26. Compute the Hessian (second derivatives), then find and classify the critical point for the function $z = x^2 - xy + y^2 - 3x - 2y$.

Problem 27. Let $f(x,y) = xy$ and $x = g(t)$, $y = h(t)$. Then use the chain rule to compute the derivative of $f(g(t), h(t))$.

Problem 28. Let $z = xy - \cos(x+y)$ and $x = e^{3t}\sin t, y = e^{-t}\cos t$. By computing $x(0), y(0), x'(0), y'(0)$ use the chain rule to compute $\frac{\mathrm{d}z}{\mathrm{d}t}$ at $t = 0$.

Problem 29. Let $f(x,y) = x^2 e^{x+y}\sin(x+y)$ and $g(u,v) = u^2 e^v \sin v$. Choose u and v to be a function of x and y such that $f(x,y) = g(u(x,y), v(x,y))$. Compute the partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial g}{\partial u}, \frac{\partial g}{\partial v}$ and then use the chain rule to compute $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$.

Problem 30. Let $g(x,y) = x^2 - y^2$. Set $x = r\cos\theta$ and $y = r\sin\theta$. Use the chain rule to compute $\frac{\partial g}{\partial r}$ and $\frac{\partial g}{\partial \theta}$ in terms of r and θ.

Problem 31. Let $h(x, y, z) = xy - yz + xz$. Compute the directional derivative of h at $(1, 2, 1)$ in the direction $\langle 3/5, 0, 4/5\rangle$.

Problem 32. Let $z = f(x, y) = x^2 + x - 3xy + y^3 - 5$. Find the points (x, y) so that $f_x = f_y = 0$. What is the interpretation of these points?

Problem 33. Let $z = f(x, y) = \sqrt{4 - x^2 - y^2}$. Find a linear approximation for f about $(1, 1)$ and use it to approximate the value of $f(1.01, 0.97)$. What happens if we approximate $(0.01, -0.03)$ by using the linear approximation about $(0, 0)$?

Problem 34. Compute the tangent plane at the specified point $P(x, y, z)$ to the surfaces defined either explicitly or implicitly below.

(a) $z = -9x^2 - 3y^2,\ P(2, 1, -39)$

(b) $xy + yz + zx = 11,\ P(1, 2, 3)$

(c) $-8x - 3y - 7z = -19,\ P(1, -1, 2)$

Problem 35. Use the multi-variable chain rule to compute $\frac{df}{dt}$ where $f(x, y) = x^2 + y^2$ and $x(t) = t, y(t) = t^2$. Repeat this calculation directly by substituting $x(t), y(t)$ and differentiating $f(x(t), y(t))$. Draw a schematic of the setup and interpret the derivative.

Problem 36. Compute the iterated integral

$$\int_0^1 \int_0^2 x^2y - y^3x \,\mathrm{d}x\,\mathrm{d}y.$$

Problem 37. Let R be the region $\{(x, y) : x > 0, 2x < y < x + 1\}$. Draw a picture of this region then integrate

$$\iint_R xy + y \,\mathrm{d}x\,\mathrm{d}y$$

Problem 38. Let D be the region bounded between the curves $x = y^2$ and $x + y = 2$. Set up an integral to compute the area of this region, and then compute

$$\iint_D x\mathrm{d}x\mathrm{d}y$$

.

Problem 39. Let $h(x, y) = x^2 + y^2 + 1$ and $g(x, y) = 1 - x - y$. Compute the volume of the region bounded above by $z = h(x, y)$, below by $z = g(x, y)$ and on the sides by $x = 0$, $x = 1$, $y = 0$ and $y = 1$.

Problem 40. Compute the iterated integral

$$\int_0^{\pi/3} \int_0^x \sin(x + y)\, \mathrm{d}y\mathrm{d}x$$

Problem 41. Let $z = xy - \cos(x + y)$ and $x = e^{3t}\sin t, y = e^{-t}\cos t$. Compute $x(0), y(0), x'(0), y'(0)$ and apply the chain rule to evaluate $\frac{\mathrm{d}z}{\mathrm{d}t}$ at $t = 0$.

Problem 42. Let $h(x, y, z) = xy - yz + xz$. Compute the gradient of h and use it to find the directional derivative of h at $(1, 2, 1)$, in the direction $\langle 3/5, 0, 4/5\rangle$.

Problem 43. Let $z = x^2 - xy + y^2 - 3x - 2y$. Find and classify the critical points of this function.

Problem 44. Let $g(x, y) = x^2 - y^2$. Set $x = r\cos\theta$ and $y = r\sin\theta$. Use the chain rule to compute $\frac{\partial g}{\partial r}$ and $\frac{\partial g}{\partial \theta}$ in terms of r and θ.

Problem 45. By converting to polar coordinates, compute the double integral $I_1 = \iint_{\mathbb{R}^2} e^{-x^2-y^2}\,\mathrm{d}x\mathrm{d}y$.

Use the fact that $e^{-x^2-y^2} = e^{-x^2}e^{-y^2}$ to relate I_1 and the Gaussian integral $I_2 = \int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x$ and thus compute I_2.

Use the same strategy to compute $\int_{-\infty}^{\infty} e^{-sx^2}\,\mathrm{d}x$ for any $s > 0$.

Problem 46. Let C be the right cone with base centered at the origin of radius 5, and of height 4. Using cylindrical polar coordinates compute

$$\iiint_C z\,\mathrm{d}x\mathrm{d}y\mathrm{d}z.$$

Do you know what this quantity represents physically?

Problem 47. Find the average distance between a point inside a sphere of radius a and the center.

Problem 48. Write the surface $z = \sqrt{3x^2 + 3y^2}$ in cylindrical coordinates. Then find the volume bounded between this surface and the surface $z = 6 - x^2 - y^2$

Problem 49. Let O be the *octant* of a sphere of radius 2 where $x > 0$, $y > 0$ and $z > 0$. Compute the integral

$$\iiint_O z^2 \,\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

using spherical coordinates.

Problem 50. Sketch the following vectors field (on different graphs): $\mathbf{v} = \langle x, -2y \rangle$, $\mathbf{w} = \langle -y, x \rangle$, $\mathbf{f} = y\mathbf{i}$.

Problem 51. Sketch the level curves of $f(x, y) = x^2 + \frac{1}{2}(y-1)^2$ and the gradient vector field ∇f.

Problem 52. By computing the mixed partials, check which of the following vector fields could be the gradient of some scalar function. If it is, then find one such scalar function : $\langle 3x^2 + y, x \rangle$, $\langle 2y, -x^2 \rangle$, $3x\mathbf{i} + 3y\mathbf{j}$, $\frac{y}{x^2+y^2}\mathbf{i} - \frac{x}{x^2+y^2}\mathbf{j}$.

Problem 53. Compute the line integral $\int_C \left(x^2 + \frac{y^2}{2}\right) \mathrm{d}S$ for C the quarter circle connecting $(2, 0)$ and $(0, 2)$.

Problem 54. Compute the line integral $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ for $\mathbf{F} = (1 + x)\mathbf{i} + (1 + y)\mathbf{j}$ along C the unit circle parametrized counter-clockwise.

Problem 55. Compute the line integral $\int_C x\mathrm{d}S$ for C the arc of the parabola $y = x^2$ between $(0, 0)$ and $(2, 4)$.

Problem 56. Compute the line integral $\int_C y\mathrm{d}x + x^2\mathrm{d}y$ for C the line segment between $(0, 0)$ and $(3, 2)$.

Problem 57. Sketch the following vectors field (on different graphs): $\mathbf{v} = \langle x, -2y\rangle$, $\mathbf{w} = \langle -y, x\rangle$, $\mathbf{f} = y\mathbf{i}$.

Problem 58. Sketch the level curves of $f(x, y) = x^2 + \frac{1}{2}(y-1)^2$ and the gradient vector field ∇f.

Problem 59. By computing the mixed partials, check which of the following vector fields could be the gradient of some scalar function. If it is, then find one such scalar function : $\langle 3x^2 + y, x\rangle$, $\langle 2y, -x^2\rangle$, $3x\mathbf{i} + 3y\mathbf{j}$, $\frac{y}{x^2+y^2}\mathbf{i} - \frac{x}{x^2+y^2}\mathbf{j}$.

Problem 60. Find a potential function for the three dimensional vector field $\langle x^3 y, \frac{x^4}{4} + 2z, 2y\rangle$.

Problem 61. Use the flux form of Green's theorem to find the flux of $\mathbf{F} = \langle x, y + x^2 \rangle$ through the triangle with vertices $(0, 0)$, $(0, 3)$ and $(1, 0)$.

Problem 62. Use Green's theorem to find the area of the region bounded by the curve $\mathbf{r}(t) = \langle \sin t \cos t, \cos t \rangle$ for $-\pi/2 \leq t \leq \pi/2$.

Problem 63. Compute the line integral $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ for $\mathbf{F} = \langle 5x + 2y, x + 3y \rangle$ for C the rectangle with vertices $(0, 1)$, $(3, 1)$, $(3, 4)$ and $(0, 4)$ traversed in that order.

Problem 64. Find the potential functions for the three dimensional vector fields $\langle x^3 y, \frac{x^4}{4} + 2z, 2y\rangle$ and $\langle -2x\sin(z)e^{-x^2-y^2}, -2y\sin(z)\exp^{-x^2-y^2}, \cos(z)e^{-x^2-y^2}\rangle$.

Problem 65. Use the flux form of Green's theorem to find the flux of $\mathbf{F} = \langle x, y + x^2\rangle$ through the triangle with vertices $(0,0)$, $(0,3)$ and $(1,0)$.

Problem 66. Compute the line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$ for $\mathbf{F} = \langle 5x + 2y, x + 3y\rangle$ for C the rectangle with vertices $(0,1)$, $(3,1)$, $(3,4)$ and $(0,4)$ traversed in that order.

Problem 67. Use Green's theorem to find the area of the region bounded by the curve $\mathbf{r}(t) = \langle \sin t \cos t, \cos t \rangle$ for $-\pi/2 \leq t \leq \pi/2$.

Problem 68. Compute the divergence of vector fields

(a) $\mathbf{F} = \langle x^2 + y^2, y^2 + z^2, x^2 + y^2 \rangle$

(b) $\mathbf{G} = \langle xe^x, ye^y, ze^z \rangle$

(c) $\mathbf{H} = \langle x^2 y, y^2 + x \rangle$.

Problem 69. Use the curl to check if the vector field $\mathbf{v} = \langle yz, xz, xy \rangle$ is conservative.

Problem 70. Describe the surface Σ given by the parametric formula $\mathbf{r}(u, v) = \langle u \cos v, u \sin v, u^2 \rangle$ for $0 \leq v < 2\pi$ and $0 \leq u \leq 2$. Compute the partials $\mathbf{r}_u$ and $\mathbf{r}_v$ and then their cross product $\mathbf{r}_u \times \mathbf{r}_v$. Use this to to compute the surface integral

$\int_\Sigma \mathbf{F} \cdot \mathrm{d}\mathbf{S}$ for $\mathbf{F} = \langle x, y, 1 \rangle$.

Compute also the surface integral $\int_\Sigma \sqrt{x^2 + y^2} \mathrm{d}S$.

Problem 71. Compute the curl of the vector fields

(a) $\mathbf{R} = \langle x + y, -x, 3y - z \rangle$

(b) $\mathbf{S} = \langle x^3 + y, -y + xz, x^2 \rangle$.

(c) Now compute the divergence of the curl $\nabla \cdot (\nabla \times \mathbf{S})$.

Problem 72. Using geometrical reasoning (without parameterizing the surface and doing the integral explicitly) compute flux $\int_\Sigma \mathbf{F} \cdot \mathrm{d}\mathbf{S}$ for $F = \langle 3x, 3y, 3z \rangle$ and Σ the sphere of radius 2 centered on the origin.

Problem 73. Compute the curl of the vector field

(a) $\mathbf{S} = \langle x^3 + y, -y + xz, x^2 \rangle$.

(b) Now compute the divergence of the curl $\nabla \cdot (\nabla \times \mathbf{S})$.

Problem 74. Find the area of the surface given by the equation $x^2 + y^2 = z^6$ with $0 \leq z \leq 1$.

Problem 75. Use Stokes theorem to compute the surface integral $\int \mathbf{F} \cdot \mathbf{N} \mathrm{d}S$ along the paraboloid $z = 4 - x^2 - y^2$, with $z > 0$ for $\mathbf{F} = \langle 0, 0, 2 \rangle$. [Hint : If $\mathbf{A} = \langle -y, x, 0 \rangle$ then $\nabla \times \mathbf{A} = \mathbf{F}$]

Problem 76. Let S be the quarter cylinder determined by the equation $x^2 + y^2 = 4$ and inequalities $x > 0$, $y > 0$, $-1 < z < 1$. Let $\mathbf{F} = \langle x, y, z^2 \rangle$ and compute the flux integral

$$\int \mathbf{F} \cdot \mathrm{d}\mathbf{S}$$

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Problem 77. Use a surface integral to find the height of the center of mass of a hemispherical shell of unit radius centered at the origin with $z > 0$. [Hint: Compute $\int z \, \mathrm{d}S$ and divide by the surface area]

Problem 78. Use the divergence theorem to find the flux $\int \mathbf{F} \cdot \mathbf{N} \mathrm{d}S$ flowing out of the rectangular box $0 \leq x \leq 3$, $1 \leq y \leq 2$, $0 \leq z \leq 1$ for the vector field $\mathbf{F} = \langle x^2 + y, x + y, z^3 + 3z \rangle$.